A NEW EFFECTIVE PRECONDITIONED METHOD
FOR L-MATRICES

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Abstract

In 1991 A.D. Gunawardena et al. reported that the convergence rate of the Gauss-Seidel method with a preconditioning matrix $I + S$ is superior to that of the basic iterative method. In this paper, we use the preconditioning matrix $I + \hat{S}$. If a coefficient matrix $A$ is a nonsingular $L$-matrix, the modified method yields considerable improvement in the rate of convergence for the iterative method. Finally, a numerical example shows the advantage of this method.

1. Introduction

For solving the linear system

$$Ax = b, \quad A = (a_{ij}) \in R^{n \times n} \text{ nonsingular and } b \in R^n.$$ (1.1)

Let $A = M - N$, and $M$ is nonsingular. Then the basic iterative scheme for Eq. (1.1) is

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\[ x_{k+1} = Tx_k + c, \quad k = 0, 1, \ldots, \]

where \( T = M^{-1}N, c = M^{-1}b \). Assuming \( A = I - L - U \), where \( I \) is the identity matrix, and \( -L \) and \( -U \) are strictly lower and strictly upper triangular parts of \( A \), respectively. \( M = I - L, N = U \) leads to the classical Gauss-Seidel method. Then the iteration matrices of the classical Jacobi and classical Gauss-Seidel methods are \( J = L + U \) and \( T = (I - L)^{-1}U \), respectively. We consider a preconditioned system of (1.1)

\[ PAx = Pb. \quad (1.2) \]

The corresponding basic iterative method is given in general by

\[ x^{(k+1)} = M_p^{-1}N_p x^{(k)} + M_p^{-1}Pb, \quad k = 0, 1, \ldots, \]

where \( PA = M_p - N_p \) is a splitting of \( PA \).

Some techniques of preconditioning (see [3, 4, 6]) which improve the rate of convergence of the classical iterative methods have been developed. For example, a simple modified Gauss-Seidel (MGS) method was first proposed by Gunawardena et al. (see [1]) with \( P = I + S \) and

\[ S = (s_{ij}) = \begin{cases} -a_{ij}, & i = j - 1, \\ 0, & i \neq j - 1. \end{cases} \quad (1.3) \]

The performance of this method on some matrices is investigated in [1].

Kohno et al. (see [3]) extend Gunawardena et al.’s work to a more general case and presented a new MGS method by using the preconditioned matrix \( P = I + S_a \),

where

\[ S_a = \begin{cases} -a_i a_{ij}, & i = j - 1, \\ 0, & i \neq j - 1 \end{cases}, \quad 0 \leq a_i \leq 1. \]

The performance of this method on some matrices is investigated in [3]. Later, Li et al. (see [5]) studied convergence of this method on \( Z \)-matrices.
In this paper, we propose another preconditioned form for MGS and Jacobi methods on $L$-matrices. Theorem 3.1 and Theorem 3.2 provide convergence results, respectively.

2. Preliminaries

For $A \in R^{n \times n}$, the directed graph $G(A)$ of $A$ is defined to be the pair $(V, E)$, where $V = \{1, 2, \cdots, n\}$ is the set of vertices and $E = \{(i, j) | a_{ij} \neq 0, 1 \leq i, j \leq n\}$ is the set of edges. A path from $i_1$ to $i_p$ is an ordered tuple of vertices $(i_1, i_2, \cdots, i_p)$ such that for each $k, (i_k, i_{k+1}) \in E$. A directed graph is said to be strongly connected if for each pair $(i, j)$ of vertices, there is a path from $i$ to $j$. The reflexive transitive closure of the graph $G(A)$ is a graph denoted by $\overline{G(A)}$. It is the smallest reflexive and transitive relation which includes the relation $G(A)$. The matrix $A$ is said to be irreducible if $\overline{G(A)}$ is strongly connected (see [5]). A matrix $A$ is called an $L$-matrix if $a_{ij} \leq 0$ for all $i, j$ such that $i \neq j$ and $a_{ii} > 0$ for all $i$ (see [2]). A matrix $A$ is said to be nonnegative if each entry of $A$ is nonnegative, and is said to be a positive matrix if each entry is positive. We shall denote this by $A \geq 0$ and $A > 0$, respectively. The spectral radius of $A$ is denoted by $\rho(A)$.

Now let us summarize the modified Gauss-Seidel method [1] with the preconditioner $P = I + S$. Let all elements $a_{ii+1}$ of $S$ be nonzero. Then we have

$$\tilde{A}x = (I + S)Ax = [I - L - SL - (U - S + SU)]x,$$

$$\tilde{b} = (I + S)b.$$  \hspace{1cm} (2.1)

Whenever

$$a_{i_{i+1}a_{i+1i}} \neq 1 \text{ for } i = 1, 2, \cdots, n - 1.$$

$(I - SL - L)^{-1}$ exists, and hence it is possible to define the Gauss-Seidel iteration matrix for $A$, namely
This iteration matrix $\tilde{T}$ is called the modified Gauss-Seidel iteration matrix.

In what follows, when $A = (a_{ij})$ is an $L$-matrix we shall assume that $a_{ii+1}a_{i+1i} > 0$ for $i = 1, 2, \cdots, n - 1$.

We next propose a new preconditioned matrix $P = I + \hat{S}$, where $\hat{S}$ is

$$\hat{S} = (\hat{s}_{ij}) = \begin{cases} -a_{ij}, & i = j + 1, \\ 0, & i \neq j + 1. \end{cases}$$

(2.3)

Let all elements $a_{i+1i}$ of $\hat{S}$ be nonzero. Thus we obtain

$$\hat{A}x = (I + \hat{S})Ax = [I - \hat{S}L - (U - \hat{S} + \hat{S}U)]x,$$

$$\hat{b} = (I + \hat{S})b.$$  

(2.4)

Whenever

$$a_{i+1i} \neq -1 \quad \text{for } i = 1, 2, \cdots, n - 1,$$

$(I - \hat{S}L - L)^{-1}$ exists, and hence it is possible to define the Gauss-Seidel iteration matrix for $\hat{A}$, namely

$$\hat{T} = (I - \hat{S}L - L)^{-1}(U - \hat{S} + \hat{S}U).$$

(2.5)

**Remark 1.** The usual splitting of $A$ is $A = D - L - U$, where $D$, $L$, and $-U$ are the diagonal, strictly lower, and strictly upper triangular parts of $A$. Since $A$ is an $L$-matrix, let us consider the new matrix $\hat{A} = D^{-1}A = I - D^{-1}L - D^{-1}U$. Clearly the system $\hat{A}x = \hat{b}$ is equivalent to the system $Ax = b$. Without loss of generality we may assume $A$ has the splitting of the form $A = I - L - U$ when $A$ is an $L$-matrix.

**Remark 2.** As the standard Gauss-Seidel iteration matrix $T = (I - L)^{-1}U$, the first column of (2.5) are zero when $a_{12} \neq a_{21}$ (In
additional, the first two columns of (2.5) are zero when $a_{12} = a_{21}$. Thus we may partition $T$ and $\hat{T}$ so that

$$T = \begin{pmatrix} 0 & T_0 \\ 0 & T_1 \end{pmatrix} \text{ and } \hat{T} = \begin{pmatrix} 0 & \hat{T}_0 \\ 0 & \hat{T}_1 \end{pmatrix},$$

(2.6)

where $T_1$ and $\hat{T}_1$ are both $(n - 1) \times (n - 1)$ matrices.

**Theorem 2.1** (Perron-Frobenius) (see [2]).

(a) If $A$ is a positive matrix, then $\rho(A)$ is a simple eigenvalue of $A$.

(b) If $A$ is nonnegative and irreducible, then $\rho(A)$ is a simple eigenvalue of $A$. Furthermore, any eigenvalue with the same modulus as $\rho(A)$ is also simple, and $A$ has a positive eigenvector $x$ corresponding to the eigenvalue $\rho(A)$. Any other positive eigenvector of $A$ is a multiple of $x$.

**Theorem 2.2** (see [5]). Let $A$ be a nonnegative matrix. Then:

(a) If $\alpha x \leq Ax$ for some nonnegative vector $x$, $x \neq 0$, then $\alpha \leq \rho(A)$.

(b) If $Ax \leq \beta x$ for some positive vector $x$, then $\rho(A) \leq \beta$. Moreover, if $A$ is irreducible and if

$$0 \neq \alpha x \leq Ax \leq \beta x$$

for some nonnegative vector $x$, then $\alpha \leq \rho(A) \leq \beta$ and $x$ is a positive vector.

**Theorem 2.3** (see [5]). Let $A$ be irreducible. If $M$ is the maximum row sum of $A$ and $m$ is the minimum row sum of $A$, then $m \leq \rho(A) \leq M$.

**Theorem 2.4** (see [5]). Let $A = I - L - U$ be an $L$-matrix with usual splitting. Let $T = (I - L)^{-1}U$ and $J = L + U$ be the Gauss-Seidel and Jacobi iteration matrices of $A$, respectively. Then exactly one of the following holds:

(a) $\rho(T) = \rho(J) = 0$,

(b) $0 < \rho(T) < \rho(J) < 1$, 
Lemma 2.5. Let $A = I - L - U$ be an $L$-matrix with usual splitting. Let $T = (I - L)^{-1}U$ and $J = L + U$ be the Gauss-Seidel and Jacobi iteration matrices of $A$, respectively. If $a_{i+1,i} \neq -1$, then $T_1$ and $\hat{T}_1$ are irreducible matrices (where $T_1$ and $\hat{T}_1$ are defined as in (2.6)).

Proof. We will show that $T_1$ is irreducible by showing that $(2, 3, \ldots, n, 2)$ is a cycle in $G(T)$; consequently it is also a cycle in $G(T_1)$, so $G(T_1)$ is strongly connected. Since $L$ is a nonnegative nilpotent matrix, $\rho(L) = 0$. Thus

$$(I - L)^{-1} = I + L + L^2 + \cdots + L^{n-1}.$$ 

Therefore $T = (I + L + L^2 + \cdots + L^{n-1})U$ and

$$G(T) = \overline{G(L)}G(U).$$

The condition $a_{i+1,i}a_{i+1,i} > 0$ implies that $(1, 2, \cdots, n)$ is a path in $G(U)$ and $(n, n-1, \cdots, 1)$ is a path in $G(L)$. Also $G(U) \subseteq G(T)$, since

$$G(T) = G(U) \cup G(LU) \cup G(L^2U) \cup \cdots \cup G(L^{n-1}U).$$

In particular, $(2, 3, \cdots, n)$ is a path in $G(T_1)$, so we only need to show that $(n, 2)$ is an edge in $G(T)$. For this note that $(n, 1)$ is an edge in $\overline{G(L)}$ and $(1, 2)$ is an edge in $G(U)$. Consequently $(n, 2)$ is an edge in $G(T)$.

Similarly, it can be shown that $\hat{T}_1$ is irreducible.

Lemma 2.6. Let $A$ be an $L$-matrix such that $a_{i+1,i} \neq -1$ for $i = 1, 2, \cdots, n - 1$. Then $\rho(T) = 1$ implies $\rho(\hat{T}) = 1$. (All matrices considered here are defined as in Remark 1.)
Proof. Clearly \( \rho(T) = 1 \) implies \( \rho(T_1) = 1 \). From Lemma 2.5, \( T_1 \) is irreducible. By Theorem 2.1 there exists a positive vector \( \omega' \) such that \( T_1\omega' = \omega' \). Now define

\[
\omega = \begin{pmatrix} T_0 \omega' \\ \omega' \end{pmatrix}.
\]

Then clearly \( T_0 \omega = \omega \). Since \( T_0 \neq 0 \), \( T_0 \omega' \) is a positive scalar and hence \( \omega \) is a positive vector. Consider

\[
\hat{T}_0 = (I - L - SL)^{-1}(U - S + SU)\omega.
\]

By factoring \( I - L \) from the right hand side we get

\[
\hat{T}_0 = [I - (I - L)^{-1}SL]^{-1}(I - L)^{-1}(U - S + SU)\omega \\
= [I - (I - L)^{-1}SL]^{-1}[(I - L)^{-1}U\omega - (I - L)^{-1}S\omega + (I - L)^{-1}SU\omega].
\]

Now \( (I - L)^{-1}U\omega = \omega \) implies \( U\omega = (I - L)\omega \). Therefore we get

\[
\hat{T}_0 = [I - (I - L)^{-1}SL]^{-1}[\omega - (I - L)^{-1}S\omega + (I - L)^{-1}S(I - L)\omega] \\
= [I - (I - L)^{-1}SL]^{-1}[\omega - (I - L)^{-1}SL\omega] \\
= [I - (I - L)^{-1}SL]^{-1}[I - (I - L)^{-1}SL\omega] \\
= \omega.
\]

Thus by Theorem 2.2, \( \rho(\hat{T}) = 1 \).

Lemma 2.7. Suppose \( A = I - L - U \) is an L-matrix such that \( a_{i+1} \neq -1 \), where \( -L \) and \( -U \) are the strictly lower and strictly upper triangular parts of \( A \), respectively. Then the standard Jacobi iteration matrix \( J = L + U \) and the modified Jacobi iteration matrix \( \hat{T} = (I - SL)^{-1}(L + U - \hat{S} + \hat{SU}) \) are both irreducible.

Proof. It follows from the condition \( a_{i+1}a_{ii+1} > 0 \) that \( (1, 2, \cdots, n - 1, n, n - 1, \cdots, 2, 1) \) is a path in \( G(L + U) \). Hence \( G(J) \) is strongly connected, and so \( J \) is irreducible.
Next we show that \( J^\hat{} \) is an irreducible matrix. Note that
\[
G(J^\hat{}) \supseteq G(L + U - \hat{S} + \hat{SU}) = G(L) \cup G(U - \hat{S}) \cup G(\hat{SU}),
\]
where the first line follows from the fact that \( (I - \hat{SL})^{-1} \) is a nonnegative matrix with a positive diagonal. Recall that \( G(L) \) contains the path \((n, n - 1, \ldots, 1)\), and note that \( G(\hat{SU}) \) contains the edges \((1, 3), (2, 4), \ldots, (n - 2, n)\). Hence \( G(J^\hat{}) \) is strongly connected and \( J^\hat{} \) is irreducible.

3. Main Results on \( L \)-matrices

**Theorem 3.1.** Suppose \( A = I - L - U \) is an \( L \)-matrix such that \( a_{i+1, i} \neq -1 \), where \(- L \) and \(- U \) are the strictly lower and strictly upper triangular parts of \( A \), respectively. Let \( T = (I - L)^{-1}U \) and \( \hat{T} = (I - \hat{SL} - L)^{-1}(U - \hat{S} + \hat{SU}) \) be the classical and modified Gauss-Seidel iteration matrices, respectively. Then

(a) \( \rho(\hat{T}) < \rho(T) \) if \( \rho(T) < 1 \),

(b) \( \rho(\hat{T}) = \rho(T) \) if \( \rho(T) = 1 \),

(c) \( \rho(\hat{T}) > \rho(T) \) if \( \rho(T) > 1 \).

**Proof.** Part (b) follows from Lemma 2.6. Now we prove (a) and (c), first we note there exists a positive vector \( \omega \) such that
\[
T\omega = \lambda\omega, \tag{3.1}
\]
where \( \lambda = \rho(T) \). Now consider
\[
\hat{T}\omega = (I - L - \hat{SL})^{-1}(U - \hat{S} + \hat{SU})\omega = (I - L - \hat{SL})^{-1}(I + \hat{S})U\omega - (I - L - \hat{SL})^{-1}\hat{S}\omega
\]
\[
(I - L - \hat{S}L)^{-1}(I + \hat{S})\lambda(I - L)\omega - (I - L - \hat{S}L)^{-1}\hat{S}\omega
\]

by (3.1).

Therefore
\[
\hat{T}_\omega - T_\omega = (I - L - \hat{S}L)^{-1}[\lambda(I + \hat{S})(I - L)\omega - \hat{S}\omega - (I - L - \hat{S}L)(I - L)^{-1}U\omega]
\]
\[
= (I - L - \hat{S}L)^{-1}[\lambda(I - L)\omega + (\lambda - 1)\hat{S}\omega - U\omega]
\]
\[
= (I - L - \hat{S}L)^{-1}(\lambda - 1)\hat{S}\omega.
\]

Write \((I - L - \hat{S}L)^{-1} = D + L'\) for some positive diagonal matrix \(D\) and a nonnegative strictly lower triangular matrix \(L'\). Then \((I - L - \hat{S}L)^{-1}\hat{S}\omega = (D + L')\hat{S}\omega \geq 0\), since \(D\hat{S}\omega \geq 0\). Also, since \(D\hat{S}\omega \neq 0\), \((I - L - \hat{S}L)^{-1}\hat{S}\omega\) is a nonzero, nonnegative vector.

If \(\lambda < 1\), then \(\hat{T}_\omega - T_\omega \leq 0\). Therefore
\[
\hat{T}_\omega \leq \omega \lambda.
\]

By using the partitioned form of
\[
\hat{T} = \begin{pmatrix} 0 & \hat{T}_0 \\ 0 & \hat{T}_1 \end{pmatrix}
\]
introduced in Remark 2, we get \(\rho(\hat{T}) < \lambda\), by Theorem 2.2.

If \(\lambda > 1\), then \(\hat{T}_\omega - T_\omega \geq 0\) but not equal to 0. Hence \(\hat{T}_\omega \leq \omega \lambda\), and this implies \(\hat{T}_1\omega_1 \geq \omega_1 \lambda\), where
\[
\omega = \begin{pmatrix} \alpha \\ \omega_1 \end{pmatrix}, \quad \omega_1 > 0.
\]

Therefore \(\rho(\hat{T}_1) > \lambda\) by Theorem 2.2. Hence \(\rho(\hat{T}) > \lambda\).

**Theorem 3.2.** Let \(A = I - L - U\) be an \(L\)-matrix, where \(-L\) and \(-U\) are the strictly lower and strictly upper triangular parts of \(A\) respectively. Let \(J = L + U\) and \(\hat{J} = (I - \hat{S}L)^{-1}(L + U - \hat{S} + \hat{S}U)\) be the classical and
modified Jacobi iteration matrices, respectively. Further assume that $J$ and $\hat{J}$ are irreducible matrices. Then

(a) $\rho(\hat{J}) < \rho(J)$ if $\rho(J) < 1,$

(b) $\rho(\hat{J}) = \rho(J)$ if $\rho(J) = 1,$

(c) $\rho(\hat{J}) > \rho(J)$ if $\rho(J) > 1.$

**Proof.** Since $J$ is irreducible, by Theorem 2.1 there exists a positive vector $\omega$ such that $J\omega = \lambda \omega$, where $\lambda = \rho(J)$. This implies

$$\omega = (L + U)\omega = \lambda \omega.$$  \hspace{1cm} (3.2)

Now consider

$$\hat{J}\omega - J\omega = (I - \hat{S}L)^{-1}[L + U - \hat{S} + \hat{S}U - (I - \hat{S}L)(L + U)]\omega$$

$$= (I - \hat{S}L)^{-1}\hat{S}[ - I + U + \hat{S} + L](L + U)\omega$$

$$= (I - \hat{S}L)^{-1}\hat{S}[ - I + U + \lambda L]\omega \quad \text{by (3.2)}$$

$$= (I - \hat{S}L)^{-1}\hat{S}( - \omega + \lambda \omega - L\omega + \lambda L\omega)$$

$$= (I - \hat{S}L)^{-1}\hat{S}(I + L)(\lambda - 1)\omega.$$  \hspace{1cm} (3.3)

If $\rho(J) < 1$, then $\hat{T}\omega \leq \rho(J)\omega$ by (3.3), and so, from Theorem 2.2, we obtain $\rho(\hat{J}) < \rho(J)$.

Similarly we can get $\rho(\hat{J}) > \rho(J)$ if $\rho(J) > 1$, and (b) also follows from Theorem 2.2.

**Corollary 3.3.** Let $A = I - L - U$, $J$, $\hat{J}$ be as definite in Theorem 3.2. Replace $J$, $\hat{J}$ are irreducible matrices by the condition $a_{i+1} \neq -1$. Then the conclusion of Theorem 3.2 holds.

**Remark.** We have tested many examples with random entries. Finally, we present some matrices to show: the modified iteration matrix
\( \hat{T} \) has a faster convergence rate when the iteration matrix \( T \) and \( \bar{T} \) are both convergent, and the modified iteration matrix \( \tilde{T} \) diverges even faster when the iteration matrix \( T \) and \( \bar{T} \) are both divergent.

The proof follows from Lemma 2.7.

4. Numerical Example

We present some matrices to illustrate our results. Let

\[
A = \begin{pmatrix} 1 & -0.2 & -0.1 & -0.4 & -0.2 \\ -0.2 & 1 & -0.3 & -0.1 & -0.6 \\ -0.3 & -0.2 & 1 & -0.1 & -0.6 \\ -0.1 & -0.1 & -0.1 & 1 & -0.01 \\ -0.2 & -0.3 & -0.4 & -0.3 & 1 \end{pmatrix} = I - L - U,
\]

where \(-L\) and \(-U\) are the strictly lower and strictly upper triangular parts of \( A \), respectively. If

\[
T = (I - L)^{-1}U, \quad \bar{T} \quad \text{and} \quad \tilde{T} \quad \text{are defined as (2.2) and (2.5), respectively.}
\]

Then \( \rho(T) = 0.9611 \), \( \rho(\bar{T}) = 0.9505 \) and \( \rho(\tilde{T}) = 0.703 \).

Let

\[
A = \begin{pmatrix} 1 & -0.0089 & -0.1305 & -0.0679 & -0.0252 \\ -0.2891 & 1 & -0.4724 & -0.2938 & -0.3628 \\ -0.1424 & -0.3383 & 1 & -0.0972 & -0.0290 \\ -0.3454 & -0.3384 & -0.4843 & 1 & -0.2982 \\ -0.0363 & -0.1415 & -0.33680 & -0.1266 & 1 \end{pmatrix} = I - L - U,
\]

then \( \rho(T) = 0.6897 \), \( \rho(\bar{T}) = 0.5610 \) and \( \rho(\tilde{T}) = 0.3832 \).
References


